NEWS AND LETTERS

73rd Annual William Lowell Putnam Mathematical Competition

Editor's Note: Additional solutions will be printed in the *Monthly* later in the year.

PROBLEMS

A1. Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

A2. Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

A3. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that

(i) $f(x) = \dfrac{2 - x^2}{2} f\left(\dfrac{x^2}{2 - x^2}\right)$ for every x in $[-1, 1]$,

(ii) $f(0) = 1$, and

(iii) $\displaystyle\lim_{x \to 1^-} \frac{f(x)}{\sqrt{1 - x}}$ exists and is finite.

Prove that f is unique, and express $f(x)$ in closed form.

A4. Let q and r be integers with $q > 0$, and let A and B be intervals on the real line. Let T be the set of all $b + mq$ where b and m are integers with b in B, and let S be the set of all integers a in A such that ra is in T. Show that if the product of the lengths of A and B is less than q, then S is the intersection of A with some arithmetic progression.

A5. Let $\mathbb{F}_p$ denote the field of integers modulo a prime p, and let n be a positive integer. Let v be a fixed vector in $\mathbb{F}_p^n$, let M be an $n \times n$ matrix with entries in $\mathbb{F}_p$, and define $G : \mathbb{F}_p^n \to \mathbb{F}_p^n$ by $G(x) = v + Mx$. Let $G^{(k)}$ denote the k-fold composition of G with itself, that is, $G^{(1)}(x) = G(x)$ and $G^{(k+1)}(x) = G(G^{(k)}(x))$. Determine all pairs p, n for which there exist v and M such that the p^n vectors $G^{(k)}(0)$, $k = 1, 2, \ldots, p^n$, are distinct.

A6. Let $f(x, y)$ be a continuous, real-valued function on $\mathbb{R}^2$. Suppose that, for every rectangular region R of area 1, the double integral of $f(x, y)$ over R equals 0. Must $f(x, y)$ be identically 0?

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B1. Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, then the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

B2. Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.

B3. A round-robin tournament among $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

B4. Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$. Does $a_n - \log n$ have a finite limit as $n \to \infty$? (Here $\log n = \log_e n = \ln n$.)

B5. Prove that, for any two bounded functions $g_1, g_2 : \mathbb{R} \to [1, \infty)$, there exist functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that, for every $x \in \mathbb{R}$,

$$\sup_{s \in \mathbb{R}} \left(g_1(s)^x g_2(s)\right) = \max_{t \in \mathbb{R}} \left(x h_1(t) + h_2(t)\right).$$

B6. Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.

SOLUTIONS

Solution to A1. Without loss of generality, assume that the d_i are in nondecreasing order. We then need $d_{i+2}^2 < d_{i+1}^2 + d_i^2$ for some i. If $d_3^2 \geq d_1^2 + d_2^2$, then $d_3^2 \geq 2d_1^2$. If in addition $d_4^2 \geq d_3^2 + d_2^2$, then $d_4^2 \geq 3d_1^2 = F_4 d_1^2$, where F_i denotes the ith Fibonacci number. By induction, either we succeed, or $d_i^2 \geq F_i d_1^2$. But $F_{12} = 144$, $d_{12} < 12$, and $d_1 > 1$, so we must succeed at some point.

Solution to A2. Assume that $a * c = b * c$, and let $e_a, d \in S$ satisfy $a * e_a = a$ and $c * d = e_a$. Then

$$a = a * e_a = a * (c * d) = (a * c) * d = (b * c) * d = b * (c * d) = b * e_a.$$

Repeating the steps so far with a and b interchanged, there exists $e_b \in S$ such that $a * e_b = b * e_b = b$. Therefore,

$$\begin{aligned} a = b * e_a &= (a * e_b) * e_a = a * (e_b * e_a) \\ &= a * (e_a * e_b) = (a * e_a) * e_b = a * e_b = b. \end{aligned}$$

Solution to A3. $f(x) = \sqrt{1-x^2}$.
Proof: On $(-1, 1)$, set $g(x) = f(x)/\sqrt{1-x^2}$. Then

$$\begin{aligned} g(x) &= \frac{(2-x^2)/2}{\sqrt{1-x^2}}\, f\left(\frac{x^2}{2-x^2}\right) \\ &= \frac{2-x^2}{2\sqrt{1-x^2}}\sqrt{1-\left(\frac{x^2}{2-x^2}\right)^2}\, g\left(\frac{x^2}{2-x^2}\right) \\ &= g\left(\frac{x^2}{2-x^2}\right). \end{aligned}$$

On $(-1, 1)$, $x^2/(2-x^2) \le |x|$ with equality only for $x = 0$, so the sequence

$$x,\ \frac{x^2}{2-x^2},\ \frac{\left(\frac{x^2}{2-x^2}\right)^2}{2-\left(\frac{x^2}{2-x^2}\right)^2},\ \ldots$$

always has limit 0. Thus, by continuity of g, $g(x) = g(0) = 1$ for all x. It follows that $f(x) = \sqrt{1-x^2}$, where the continuity of f was used to show equality at the endpoints.

Note. As the proof shows, condition (iii) is actually unnecessary. (It was left in to provide a hint of the form of the solution.)

Solution to A4. Let $a_1 < a_2 < a_3$ be consecutive terms in S. We need only show $a_2 - a_1 = a_3 - a_2$. If not, replacing A and r with $-A$ and $-r$ if necessary, we may assume $a_2 - a_1 < a_3 - a_2$. Let $b_k \in B$ such that $ra_k \equiv b_k \pmod q$, $k = 1, 2, 3$. Replacing r and B with $-r$ and $-B$ if necessary, we may assume $b_1 \le b_2$. We have

$$(a_3 - a_2)(b_2 - b_1) \equiv r(a_3 - a_2)(a_2 - a_1) \equiv (a_2 - a_1)(b_3 - b_2) \pmod q.$$

Because

$$\begin{aligned} |(a_3 - a_2)(b_2 - b_1) - (a_2 - a_1)(b_3 - b_2)| &\le (a_3 - a_2)|b_2 - b_1| + (a_2 - a_1)|b_3 - b_2| \\ &\le (a_3 - a_1)\cdot|B| \le |A|\cdot|B| < q\,, \end{aligned}$$

we have $(a_3 - a_2)(b_2 - b_1) = (a_2 - a_1)(b_3 - b_2)$ or $\dfrac{b_2 - b_1}{a_2 - a_1} = \dfrac{b_3 - b_2}{a_3 - a_2} \ge 0$, so $a_2 - a_1 < a_3 - a_2$ implies $b_2 - b_1 \le b_3 - b_2$. Then, however,

$$a_2 < 2a_2 - a_1 < a_3,\quad r(2a_2 - a_1) \equiv 2b_2 - b_1 \pmod q,$$

and

$$b_2 \le 2b_2 - b_1 \le b_3,$$

so $2b_2 - b_1$ is in the interval B and hence $2a_2 - a_1 \in S$, a contradiction.

Solution to A5. Such v and M exist for $n = 1$ and all p and for $n = 2$, $p = 2$.
For $n = 1$, set $v = [1]$ and $M = [1]$. For $p = n = 2$, set

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Conversely, suppose v and M exist. First observe that to get distinct values, the earliest possible occurrence of 0 is $G^{(p^n)}(0)$. Thus,

$$v + Mv + M^2 v + \cdots + M^{p^n - 1} v = 0.$$

Multiplying by M and combining the two expressions yields $M^{p^n} v = v$. But then, for all k,

$$M^{p^n}(v + Mv + \cdots + M^k v) = v + Mv + \cdots + M^k v.$$

Thus, M^{p^n} is the identity matrix. It follows that the minimal polynomial of M divides $x^{p^n} - 1 = (x-1)^{p^n}$. By the Cayley-Hamilton theorem, the minimal polynomial of M divides its characteristic polynomial; in particular, the minimal polynomial has degree at most n and so $(M - I)^n = 0$.

If neither $n = 1$ nor $p = n = 2$ holds, $p^{n-1} - 1 \geq n$, so $(M - I)^{p^{n-1}-1} = 0$. However,

$$(x-1)^{p^{n-1}-1} = \frac{(x-1)^{p^{n-1}}}{x-1} = \frac{x^{p^{n-1}} - 1}{x-1} = 1 + x + \cdots + x^{p^{n-1}-1}.$$

But then $G^{(p^{n-1})}(0) = 0$, a contradiction.

Solution to A6. Yes, $f(x, y)$ is identically 0, even if one only considers rectangles with sides parallel to the x- and y- axes.

For every $w > 0$ and every x, y,

$$\int_x^{x+w} \int_y^{y+1/w} f(u, v)\, dv\, du = 0.$$

Differentiating with respect to x, by the fundamental theorem of calculus we have

$$\int_y^{y+1/w} [f(x+w, v) - f(x, v)]\, dv = 0.$$

Differentiating with respect to y this time yields

$$f(x+w, y+1/w) - f(x, y+1/w) - f(x+w, y) + f(x, y) = 0.$$

Therefore, $f(x+w, y) - f(x, y)$ has period $1/w$ in y. Thus,

$$f(x+w+z, y) - f(x+z, y) - f(x+w, y) + f(x, y) \tag{1}$$

has periods $1/w$ and $1/z$ in y. Choosing w and z so that z/w is irrational, this and continuity imply (1) is independent of y, because any real y can be approximated arbitrarily closely by numbers of the form $m/w + n/z$ with m, n integers. Integrating (1) over any rectangle of the form $[x, x+\varepsilon] \times [y, y+1/\varepsilon]$ gives four terms that are each the integral of $f(x, y)$ over some shifted rectangle of area 1, so it yields 0. By taking ε sufficiently small, continuity implies that (1) is identically 0. It follows that $f(x+w, y) - f(x, y)$ has period z in x. Since we may choose any z for which z/w is irrational, $f(x+w, y) - f(x, y)$ is independent of x. As above, integrating over $[x, x+1/\varepsilon] \times [y, y+\varepsilon]$, it follows that it is identically 0. Since w is arbitrary $f(x, y)$ is independent of x. Similarly it is independent of y, hence constant, hence identically 0.

Solution to B1. By rule (ii), $f_2(f(x)) = \ln(f(x)+1) \in S$ and $\ln(g(x)+1) \in S$, so that

$$\ln(f(x)+1) + \ln(g(x)+1) = \ln\big(f(x)g(x) + f(x) + g(x) + 1\big) \in S.$$

Therefore,

$$e^{\ln\left(f(x)g(x)+f(x)+g(x)+1\right)} - 1 = f(x)g(x) + f(x) + g(x) \in S.$$

Because $f(x) + g(x) \in S$ and $f(x)g(x) + f(x) + g(x) \geq f(x) + g(x)$ for every $x \in [0, \infty)$, it follows that $f(x)g(x) \in S$.

Solution to B2. Let $F_1, \ldots, F_f$ be the faces of P, and let $\{B_1, \ldots, B_n\}$ be a collection of n balls of radii $r_1, \ldots, r_n$ respectively, such that $\cup_{i=1}^{f} F_i \subseteq \cup_{j=1}^{n} B_j$. Denote by $A(X)$ the area of a two-dimensional figure X, and by A the total surface area of P; note that

$$A = \sum_{i=1}^{f} A(F_i), \qquad V = \frac{4}{3}\pi \sum_{j=1}^{n} r_j^3.$$

Since

$$A(F_i \cap B_j) \leq \pi r_j^2,$$

it follows that

$$A = \sum_{i=1}^{f} A(F_i) \leq \sum_{i=1}^{f} \sum_{j=1}^{n} A(F_i \cap B_j) \leq \pi f \sum_{j=1}^{n} r_j^2.$$

From Hölder's inequality, we have that

$$\left(\sum_{j=1}^{n} (r_j^2)^{3/2}\right)^{2/3} \left(\sum_{j=1}^{n} 1^3\right)^{1/3} \geq \sum_{j=1}^{n} r_j^2 \geq \frac{A}{\pi f},$$

$$\left(\frac{3}{4\pi} V\right)^{2/3} n^{1/3} \geq \frac{A}{\pi f},$$

$$n \geq \frac{A^3}{\pi^3 f^3} \left(\frac{4\pi}{3V}\right)^2 = \frac{16A^3/9}{\pi f^3 V^2}.$$

We see that we can take any $c = c(P)$ with $0 < c < 16A^3/9\pi f^3$.

Solution to B3 (based on a student paper). Yes. For a proof, first consider the special case in which for all i and j with $i > j$, team i defeated team j in their encounter. Then start by choosing team 2 from the round in which teams 1 and 2 played each other, and then go through the other teams $3, 4, \ldots, n$ in order. Each of those teams, say i, has $i - 1$ victories, and when we get to team i, winners from only $i - 2$ rounds have been chosen, so it is possible to choose team i as the winner for a new round; all teams except team 1 will be chosen and no team will be chosen more than once, so we are done in this case.

It is now enough to show that if we can make a set of choices with the desired property in one tournament, then we can again do so if the outcome of a single game in the tournament is changed. We may assume that the game whose outcome is changed was between teams 1 and 2 and was originally won by team 1. If team 1 was not chosen in that round, we can keep the same choices. If team 2 was not chosen in any other round, we can choose team 2 in that round and keep all the other choices.

Otherwise, we will choose team 2 in that round, and in the round for which team 2 was originally chosen, we consider team 1's opponent, say team 3. If team 1 defeated team 3, or if team 3 was not chosen in any other round, we can choose the winner of the game between teams 1 and 3 and we will be done. Otherwise, we choose team 3 anyway, and consider team 1's opponent, say team 4, in the round for which team 3 was originally chosen. In general, if we have now chosen teams 2 through i in rounds in which they defeated team 1 and team i was originally chosen in a different round, then we label team 1's opponent in that different round as team $i+1$, and we choose the winner of the game between teams 1 and $i+1$. Eventually this process will terminate, either because team 1 defeated team $i+1$ or because team $i+1$ is the team that was originally not chosen in any round, and we are then done.

Solution to B4. Let $f(x) = x + e^{-x}$, so $a_{n+1} = f(a_n)$. Note that for $x > 0$, $f'(x) = 1 - e^{-x} > 0$, so f is increasing for $x \geq 0$. We now show by induction on n that $a_n > \log(n+1)$ for all n. The base case is clear, and $a_n > \log(n+1)$ implies

$$\begin{aligned} a_{n+1} = f(a_n) > f(\log(n+1)) &= \log(n+1) + \frac{1}{n+1} \\ &> \log(n+1) + \int_{n+1}^{n+2} \frac{1}{x}\, dx = \log(n+2), \end{aligned}$$

completing the induction.

It follows that

$$\begin{aligned} a_{n+1} - a_n = e^{-a_n} &< \frac{1}{n+1} \\ &< \int_{n}^{n+1} \frac{1}{x}\, dx = \log(n+1) - \log n, \end{aligned}$$

so $a_{n+1} - \log(n+1) < a_n - \log n$. Thus the $a_n - \log n$ form a decreasing sequence of positive numbers, so they have a limit. (It can be shown that the limit is 0.)

Solution to B5. Note that every function $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = \sup_{t \in \mathbb{R}} \big(x h_1(t) + h_2(t) \big) \tag{2}$$

is convex, where $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ are any two functions subject only to the condition that the supremum on the right-hand side exists for every $x \in \mathbb{R}$. Indeed, for every $x, y \in \mathbb{R}$ and every $\lambda \in (0, 1)$,

$$\begin{aligned} &\lambda f(x) + (1-\lambda) f(y) \\ &\quad = \sup_{t \in \mathbb{R}} \big(\lambda x h_1(t) + \lambda h_2(t) \big) + \sup_{t \in \mathbb{R}} \big((1-\lambda) y h_1(t) + (1-\lambda) h_2(t) \big) \\ &\quad \geq \sup_{t \in \mathbb{R}} \Big(\big((\lambda x + (1-\lambda) y \big) h_1(t) + \big(\lambda + (1-\lambda) \big) h_2(t) \Big) \\ &\quad = f\big(\lambda x + (1-\lambda) y \big), \end{aligned}$$

so that f is a convex function. We claim that the converse is also true, namely, every convex function $f : \mathbb{R} \to \mathbb{R}$ satisfies (2) for some $h_1, h_2 : \mathbb{R} \to \mathbb{R}$; in fact, we claim that h_1, h_2 can be chosen so that the slightly stronger condition

$$f(x) = \max_{t\in\mathbb{R}} \big(xh_1(t) + h_2(t)\big) \tag{3}$$

holds. Indeed, since f is convex, we know that f is a continuous function with left and right derivatives at every point, which satisfies

$$f'_-(a) \le f'_+(a) \le \frac{f(b) - f(a)}{b - a} \le f'_-(b)$$

for every $a, b \in \mathbb{R}$ with $a < b$. From this it follows that

$$f(x) \ge (x - t)f'_-(t) + f(t)$$

for every $t \in \mathbb{R}$, with equality for $t = x$. It follows that (3) holds with

$$h_1(t) = f'_-(t), \quad h_2(t) = f(t) - tf'_-(t).$$

Let $g_1, g_2 : \mathbb{R} \to [1, \infty)$ be as in the statement of the problem. In the first part of the above discussion, we have proved that

$$f(x) = \sup_{t\in\mathbb{R}} \big(x \log g_1(t) + \log g_2(t)\big)$$

defines a convex function $f : \mathbb{R} \to \mathbb{R}$, since $\log g_1, \log g_2 : \mathbb{R} \to \mathbb{R}$ are bounded. But then the function $e^{f(x)}$ is also convex. This is well known and follows from

$$\lambda e^{f(x)} + (1 - \lambda)e^{f(y)} \ge e^{\lambda f(x)+(1-\lambda)f(y)} \ge e^{f(\lambda x+(1-\lambda)y)},$$

where the first step uses the convexity of the exponential function, and the second step uses the convexity of f and the monotonicity of the exponential function. Since $e^{f(x)}$ is a convex function, it follows by (3) that

$$e^{f(x)} = \max_{t\in\mathbb{R}} \big(xh_1(t) + h_2(t)\big)$$

for some $h_1, h_2 : \mathbb{R} \to \mathbb{R}$. This is equivalent to what was to be proved.

Solution to B6. Consider $a \ne 0, 1, -1$, the three classes fixed by π. The cycle containing a has the same length as the cycle containing $-a \not\equiv a \pmod p$. Thus, the parity of π is determined by those cycles containing both a and $-a$. Similarly, the cycle containing a has the same length as the cycle containing $a^{-1} \not\equiv a \pmod p$. Thus we are down to cycles containing a, $-a$, and a^{-1}. Then if it takes k applications of π to get from a to $-a$, the cycle will have length $2k$; on the other hand, the same argument applies to a^{-1} instead of $-a$, so $-a \equiv a^{-1} \pmod p$, that is, $a^2 \equiv -1 \pmod p$. For such a, $a^3 \equiv -a \pmod p$ and so $k = 1$. Because the multiplicative group $\pmod p$ is cyclic of order $p - 1$, or by Euler's criterion, there are no such a when $p \equiv 3 \pmod 4$ and there are two that form a cycle of order 2 when $p \equiv 1 \pmod 4$. Therefore, π is even in the former case and odd in the latter.