# SEVENTY FOURTH ANNUAL
# WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

**Saturday, December 7, 2013** **Examination A**

## Problem A1

Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

## Problem A2

Let $S$ be the set of all positive integers that are *not* perfect squares. For $n$ in $S$, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of $a_r$ over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function $f$ from $S$ to the integers is one-to-one.

## Problem A3

Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and $x$, with $0 < x < 1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number $y$ with $0 < y < 1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

## Problem A4

A finite collection of digits 0 and 1 is written around a circle. An *arc* of length $L \geq 0$ consists of $L$ consecutive digits around the circle. For each arc $w$, let $Z(w)$ and $N(w)$ denote the number of 0's in $w$ and the number of 1's in $w$, respectively. Assume that $|Z(w) - Z(w')| \leq 1$ for any two arcs $w, w'$ of the same length. Suppose that some arcs $w_1, \ldots, w_k$ have the property that

$$Z = \frac{1}{k}\sum_{j=1}^{k} Z(w_j) \text{ and } N = \frac{1}{k}\sum_{j=1}^{k} N(w_j)$$

are both integers. Prove that there exists an arc $w$ with $Z(w) = Z$ and $N(w) = N$.

**Problem B4**

For any continuous real-valued function $f$ defined on the interval $[0,1]$, let

$$\mu(f)=\int_0^1 f(x)\,dx,\quad \mathrm{Var}(f)=\int_0^1 \left(f(x)-\mu(f)\right)^2 dx,\quad M(f)=\max_{0\le x\le 1}|f(x)|.$$

Show that if $f$ and $g$ are continuous real-valued functions defined on the interval $[0,1]$, then

$$\mathrm{Var}(fg)\le 2\,\mathrm{Var}(f)M(g)^2+2\,\mathrm{Var}(g)M(f)^2.$$

**Problem B5**

Let $X=\{1,2,\ldots,n\}$, and let $k\in X$. Show that there are exactly $k\cdot n^{n-1}$ functions $f:X\to X$ such that for every $x\in X$ there is a $j\ge 0$ such that $f^{(j)}(x)\le k$.

[Here $f^{(j)}$ denotes the $j^{\text{th}}$ iterate of $f$, so that $f^{(0)}(x)=x$ and $f^{(j+1)}(x)=f\left(f^{(j)}(x)\right)$.]

**Problem B6**

Let $n\ge 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of $n$ spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space $s$, places a stone in the nearest empty space to the left of $s$ (if such a space exists), and places a stone in the nearest empty space to the right of $s$ (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?