



SEVENTY FOURTH ANNUAL  
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 7, 2013

Examination A

**Problem A1**

Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

**Problem A2**

Let  $S$  be the set of all positive integers that are *not* perfect squares. For  $n$  in  $S$ , consider choices of integers  $a_1, a_2, \dots, a_r$  such that  $n < a_1 < a_2 < \dots < a_r$  and  $n \cdot a_1 \cdot a_2 \cdots a_r$  is a perfect square, and let  $f(n)$  be the minimum of  $a_r$  over all such choices. For example,  $2 \cdot 3 \cdot 6$  is a perfect square, while  $2 \cdot 3$ ,  $2 \cdot 4$ ,  $2 \cdot 5$ ,  $2 \cdot 3 \cdot 4$ ,  $2 \cdot 3 \cdot 5$ ,  $2 \cdot 4 \cdot 5$ , and  $2 \cdot 3 \cdot 4 \cdot 5$  are not, and so  $f(2) = 6$ . Show that the function  $f$  from  $S$  to the integers is one-to-one.

**Problem A3**

Suppose that the real numbers  $a_0, a_1, \dots, a_n$  and  $x$ , with  $0 < x < 1$ , satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \dots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number  $y$  with  $0 < y < 1$  such that

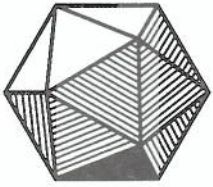
$$a_0 + a_1 y + \dots + a_n y^n = 0.$$

**Problem A4**

A finite collection of digits 0 and 1 is written around a circle. An *arc* of length  $L \geq 0$  consists of  $L$  consecutive digits around the circle. For each arc  $w$ , let  $Z(w)$  and  $N(w)$  denote the number of 0's in  $w$  and the number of 1's in  $w$ , respectively. Assume that  $|Z(w) - Z(w')| \leq 1$  for any two arcs  $w, w'$  of the same length. Suppose that some arcs  $w_1, \dots, w_k$  have the property that

$$Z = \frac{1}{k} \sum_{j=1}^k Z(w_j) \quad \text{and} \quad N = \frac{1}{k} \sum_{j=1}^k N(w_j)$$

are both integers. Prove that there exists an arc  $w$  with  $Z(w) = Z$  and  $N(w) = N$ .



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Examination B

**Problem B4**

For any continuous real-valued function  $f$  defined on the interval  $[0,1]$ , let

$$\mu(f) = \int_0^1 f(x) dx, \quad \text{Var}(f) = \int_0^1 (f(x) - \mu(f))^2 dx, \quad M(f) = \max_{0 \leq x \leq 1} |f(x)|.$$

Show that if  $f$  and  $g$  are continuous real-valued functions defined on the interval  $[0,1]$ , then

$$\text{Var}(fg) \leq 2 \text{Var}(f)M(g)^2 + 2 \text{Var}(g)M(f)^2.$$

**Problem B5**

Let  $X = \{1, 2, \dots, n\}$ , and let  $k \in X$ . Show that there are exactly  $k \cdot n^{n-1}$  functions  $f: X \rightarrow X$  such that for every  $x \in X$  there is a  $j \geq 0$  such that  $f^{(j)}(x) \leq k$ .

[Here  $f^{(j)}$  denotes the  $j^{\text{th}}$  iterate of  $f$ , so that  $f^{(0)}(x) = x$  and  $f^{(j+1)}(x) = f(f^{(j)}(x))$ .]

**Problem B6**

Let  $n \geq 1$  be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of  $n$  spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space  $s$ , places a stone in the nearest empty space to the left of  $s$  (if such a space exists), and places a stone in the nearest empty space to the right of  $s$  (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?