2014 Putnam Problem 3B

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Proposition 1. Suppose that A is an $m \times n$ matrix with rational entries such that the matrix of absolute values of A, $\widetilde{A}$, contains at least $m+n$ distinct primes. Prove that A has rank at least two.

Proof. Note that it suffices to show that not all rows of A are scalar multiples of the first row of A.

Throughout this problem, we let S denote a set of $m+n$ distinct primes in $\widetilde{A}$. First, we prove a lemma:

Lemma 2. Let G be a graph on n vertices with n edges. Then, G contains a loop.

Proof. Proceed by induction. The case $n = 1$ is trivial. Now, consider a graph G with $n+1$ vertices and n edges. By the pigeonhole principle, G must contain a vertex v of index at most one. Excise v and its edges from G. If v has index zero, remove v and any edge. What remains is a subgraph with n vertices and n edges. By our inductive hypothesis, this subgraph contains a loop. So, G contains a loop. □

To proceed, we will first solve the case when A is $n \times n$, then show that this argument effectively handles the general $m \times n$ case.

Let A be an $n \times n$ matrix satisfying the conditions of the proposition. Define a graph $G(A)$ on A as follows. Draw a vertex v_i corresponding to each column c_i for $1 \leq i \leq n$. Now, define a function on the rows of A as follows. Let

$$p : \{\text{rows of } A\} \longrightarrow \mathbf{Z}_{\geq 0}$$
$$p(r_i) = \#\{\text{elements of } S \text{ in row } i\}$$

Begin in row 1. Consider the C_1 of columns containing an element in S is row one. Draw a connected tree between the vertices corresponding to columns in C_1. Note that this tree has $p(r_1) - 1$ edges. Do the same for row j, for $2 \leq j \leq n$, adding the new edges onto edges we have already drawn. Note that each row r_i adds $p(r_i) - 1$ edges.

Decompose S as follows. For $i = 1, \ldots, n$, let S_i be the primes in S contained in row i of A. Note $S_i \cap S_j = \emptyset$ for $i \neq j$ and $S = \bigcup_{i=1}^{n} S_i$.

Lemma 3. $G(A)$ contains a loop.

Proof. We will now count the number of edges in $G(A)$, which we will call $E(G(A))$.

$$\begin{aligned} E(G(A)) &= \sum_{i=1}^{n} p(r_i) - 1 \\ &= \sum_{i=1}^{n} p(r_i) - \sum_{i=1}^{n} 1 \\ &= 2n - n \\ &= n \end{aligned}$$

It follows from lemma 2 that $G(A)$ contains a loop. □

Now, suppose that all columns of A are scalar multiples of the first colum (that A has rank at most one). Then, when two vertices v_i, v_j are connected by an edge in $G(A)$, $c_i = \lambda c_j$, where λ can be expressed as a ratio of elements of S that live in the same row of A.

Choose a vertex v_i contained in a loop of G. Then, there are scalars $\lambda_0, \lambda_1, \ldots, \lambda_k$ such that

$$\begin{aligned} c_i &= \lambda_1 \ldots \lambda_k c_j \\ c_i &= \lambda_0 c_j \end{aligned}$$

where $|\lambda_i|$ is a ratio of distinct primes in S.

We will derive a contradiction from the equation

$$|\lambda_0| = |\lambda_1 \ldots \lambda_k| \tag{1}$$

By performing an elementary row operation, we may assume without loss of generality that λ_0 is the ratio of two primes in S_1, say p_0/p_1.

Our graph was constructed so that if $\lambda_{i_1}, \ldots, \lambda_{i_r}$ are ratios of primes in S_j, then $\lambda_{i_1} \ldots \lambda_{i_r} \neq 1$. In particular, this product, in lowest terms will contain at least one primes in S_j in both its numerator and denominator.

Also, by construction, product $\lambda_1 \ldots \lambda_k$ must contain a λ_k such that λ_k is a ratio of primes in S_i for $i > 1$. This is because row 1 induced no loops in the graph. By the previous observation, one such ratio p_2/p_3 remains when $\lambda_1 \ldots \lambda_k$ is expressed in lowest terms. Clearing the denominator, we have

$$p_3 p_0 p_1^* \ldots p_s^* = p_1 p_2 p_1' \ldots p_s'$$

where the other primes are in S. This contradicts the fundamental theorem of arithmetic.

This proves the $n \times n$ case. Note that in an $m \times n$ matrix for $m > n$. Then, there must be a row that contains 1 or fewer elements of S. Perform a row operation to move this row to the bottom and consider the upper $m - 1 \times n$ submatrix. Repeat until we have an $n \times n$ matrix containing $2n$ entries and apply the preivous proof. □