Proposition 1. Suppose \( f \) is a function on the interval \([1, 3]\) such that \(-1 \leq f(x) \leq 1\) for all \(x\) and \(\int_1^3 f(x) \, dx = 0\). Then, \(\left| \int_1^3 \frac{f(x)}{x} \, dx \right| \leq \log(4/3)\) and this bound is attained.

Proof. Let \( \chi_I \) denote the characteristic function of the interval \( I \). We will first show

\[
\int_1^3 \frac{\chi_{[1,2]} - \chi_{[2,3]}}{x} \, dx = \log 4/3
\]  

(1)

We have

\[
\int_1^3 \frac{\chi_{[1,2]} - \chi_{[2,3]}}{x} \, dx = \int_1^2 \frac{1}{x} \, dx - \int_2^3 \frac{1}{x} \, dx
\]

\[
= \log 2 - \log 1 - (\log 3 - \log 2)
\]

\[
= 2 \log 2 - \log 3
\]

\[
= \log 4/3
\]

Now, we make a mono-invariance argument:

Lemma 2. Let \( h(x) \) be a function on \([1, 3]\) such that \(\int_1^3 h(x) \, dx = 0\) and suppose that \( h_1(x) := h(x)|_{[1,2]} \geq 0 \) and \( h_2(x) := h(x)|_{[2,3]} \leq 0 \). Then,

\[
\int_1^3 \frac{h(x)}{x} \, dx \geq 0
\]  

(2)

Proof.

\[
\int_1^3 \frac{h(x)}{x} \, dx = \int_1^2 \frac{h_1(x)}{x} \, dx + \int_2^3 \frac{h_2(x)}{x} \, dx
\]

\[
\geq \int_1^2 \frac{h_1(x)}{2} \, dx + \int_2^3 \frac{h_2(x)}{2} \, dx
\]

\[
\geq \frac{1}{2} \left( \int_1^2 h_1(x) \, dx + \int_2^3 h_2(x) \, dx \right)
\]

\[
\geq \int_1^3 h(x) \, dx
\]

\[
\geq 0
\]
Now, let $f$ be a function on $[1, 3]$. Let

$$g(x) = \begin{cases} 
1 - f(x) & x \in [1, 2) \\
-1 - y f(x) & x \in [2, 3]
\end{cases}$$

be a function from $[1, 3]$ to $\mathbb{R}$. Then, $g$ satisfies the conditions of lemma 2 and so $\int_1^3 \frac{g(x)}{x} \geq 0$. Consequently,

$$\int_1^3 \frac{\chi_{[1,2]} - \chi_{[2,3]}}{x} = \int_1^3 \frac{g(x) + f(x)}{x} \, dx \geq \int_1^3 \frac{f(x)}{x} \, dx$$

and so the integral is maximized by $\chi_{[1,2]} - \chi_{[2,3]}$, which gives the value $\log 4/3$. Similar arguments show that the minimum value of the integral is given by $\chi_{[2,3]} - \chi_{[1,2]}$. Here, the value of that integral is $\log 3/4 = -\log 4/3$. \qed