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Proposition 1. Suppose f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\ dx = 0$. Then, $\left|\int_1^3 \frac{f(x)}{x}\ dx\right| \leq \log(4/3)$ and this bound is attained.

Proof. Let χ_I denote the characeristic function of the interval I. We will first show

$$\int_1^3 \frac{\chi_{[1,2)} - \chi_{[2,3]}}{x} = \log 4/3 \tag{1}$$

We have

$$\begin{aligned} \int_1^3 \frac{\chi_{[1,2)} - \chi_{[2,3]}}{x} &= \int_1^2 \frac{1}{x}\ dx - \int_2^3 \frac{1}{x}\ dx \\ &= \log 2 - \log 1 - (\log 3 - \log 2) \\ &= 2\log 2 - \log 3 \\ &= \log 4/3 \end{aligned}$$

Now, we make a mono-invariance argument:

Lemma 2. Let $h(x)$ be a function on $[1,3]$ such that $\int_1^3 h(x) = 0$ and suppose that $h_1(x) := h(x)|_{[1,2)} \geq 0$ and $h_2(x) := h(x)|_{[2,3]} \leq 0$. Then,

$$\int_1^3 \frac{h(x)}{x}\ dx \geq 0 \tag{2}$$

Proof.

$$\begin{aligned} \int_1^3 \frac{h(x)}{x}\ dx &= \int_1^2 \frac{h_1(x)}{x}\ dx + \int_2^3 \frac{h_2(x)}{x}\ dx \\ &\geq \int_1^2 \frac{h_1(x)}{2}\ dx + \int_2^3 \frac{h_2(x)}{2}\ dx \\ &\geq \frac{1}{2}\left(\int_1^2 h_1(x)\ dx + \int_2^3 h_2(x)\ dx\right) \\ &\geq \int_1^3 h(x)\ dx \\ &\geq 0 \end{aligned}$$

□

Now, let f be a function on $[1,3]$. Let

$$g(x) = \begin{cases} 1 - f(x) & x \in [1,2) \\ -1 - yf(x) & x \in [2,3] \end{cases}$$

be a function from $[1,3]$ to $\mathbf{R}$. Then, g satisfies the conditions of lemma 2 and so $\int_1^3 \frac{g(x)}{x} \geq 0$. Consequently,

$$\int_1^3 \frac{\chi_{[1,2)} - \chi_{[2,3]}}{x} = \int_1^3 \frac{g(x) + f(x)}{x}\, dx \geq \int_1^3 \frac{f(x)}{x}\, dx$$

and so the integral is maximized by $\chi_{[1,2)} - \chi_{[2,3]}$, which gives the value $\log 4/3$. Similar arguments show that the minimum value of the integral is given by $\chi_{[2,3]} - \chi_{[1,2)}$. Here, the value of that integral is $\log 3/4 = -\log 4/3$. □