Solutions to Green Chicken Contest 2015

1. The number of employees is irrelevant. There are $\binom{7}{5}$ = 21 ways to group five people out of the set of seven who appear to be spies. But only one of these groupings exactly includes the five spies. So the answer is 1/21, hence the name of the company.

2. The answer is 16. (Recall that 1 is not a prime.) Note that 1 + 3 = 4, 2 + 7 = 9, 3 + 13 = 16, 4 + 5 = 9, 5 + 11 = 16, 6 + 3 = 9, 7 + 2 = 9, 8 + 17 = 25, 9 + 7 = 16, 10 + 71 = 81, 11 + 5 = 16, 12 + 13 = 25, 13 + 3 = 16, 14 + 2 = 16, 15 + 181 = 196. However, if 16 + p = s^2, then p = (s + 4)(s – 4). If p is prime, then s – 4 = 1 implying that s = 5 and (s +4)(s – 4) = 9, a contradiction.

3. (a) Recall that $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x. Hence $\sum_{n=0}^{\infty}\frac{1}{n!}$ = e.

   (b) $\sum_{n=0}^{\infty}\frac{1}{(n+1)n!} = \sum_{n=0}^{\infty}\frac{1}{(n+1)!} = \sum_{n=1}^{\infty}\frac{1}{n!}$ = e – 1.

   (c) $\sum_{n=0}^{\infty}\frac{1}{(n+2)n!} = \sum_{n=0}^{\infty}\frac{n+1}{(n+2)!} = \sum_{n=0}^{\infty}\frac{(n+2)-1}{(n+2)!} = \sum_{n=0}^{\infty}\frac{1}{(n+1)!} - \sum_{n=0}^{\infty}\frac{1}{(n+2)!}$
   = (e – 1) – (e – 2) = 1.

4. The set P looks like an ice cream cone lying on its side. Answer is $4\sqrt{3}+\frac{8\pi}{3}$.

5. Let n be a positive integer. Consider the set S = {1, 11, 111, ..., $11...11_{n+1}$} where the last number consists of n+1 1's. Since there are exactly n residue classes modulo n, by the pigeonhole principle, there are two elements of S, say $11...11_r$ and $11...11_s$ consisting of r 1's and s 1's respectively with r < s, that are congruent to each other modulo n. But then n divides $11...11_s$ – $11...11_r$ which is the number consisting of s – r 1's followed by r 0's.

   Alternatively, one can note that if gcd(n, 10) = 1, then by the Euler-Fermat Theorem (extension of Fermat's Little Theorem), $10^{\phi(n)} \equiv 1 (\bmod n)$. Hence n divides $10^{\phi(n)}-1$ which is 99...99 with $\phi(n)$ 9's. If additionally gcd(n,3) = 1, then we can divide by 9 and hence n divides 11...11 with $\phi(n)$ 1's. If 3^k exactly divides n, then just repeat the string of 1's 3^k times if need be. Furthermore, if 2^a5^b exactly divides n, then just tack on max{a, b} 0's.

6. (a) Let f(x) = 2 sin $\left(\frac{x}{2}\right)$. Since |sin $\left(\frac{x}{2}\right)$| ≤ 1 for all x, |f(x)| ≤ 2 for all x.

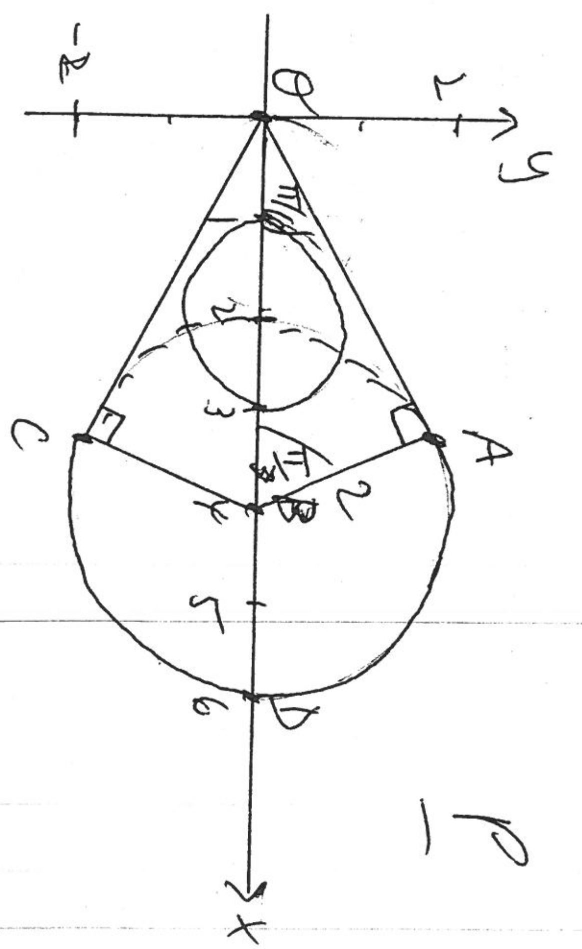
Furthermore, $f'(x) = \cos\frac{x}{2}$. Hence $f(x)f'(x) = 2\sin(\frac{x}{2})\cos(\frac{x}{2}) = \sin x$ by the double-angle formula for sine.

(b) No. Suppose otherwise for some such function f.

But $\int_0^x 2f(t)f'(t)\,dt = \left[f^2(t)\right]_0^x = f^2(x) - f^2(0)$.

By the second condition, $\int_0^x 2f(x)f'(x)\,dx \geq \int_0^x 2\sin x\,dx = -2\cos x\,\Big|_0^x$ = 2 − 2 cos(x).

But then $f^2(x) - f^2(0) \geq 2 - 2\cos(x)$. Let $x = \pi$.
Then $f^2(\pi) \geq 2 - 2\cos(\pi) + f^2(0) \geq 4$.
Hence $|f(\pi)| \geq 2$, contradicting the first condition.



$\Delta OAB \cong \Delta OCB$

$AB = 2$, $OB = 4$

$\angle BOA = \pi/6$, $\angle OAB = \pi/2$,

and hence $\angle ABO = \pi/3$

So $OB = \sqrt{16-4} = 2\sqrt{3}$

Area P = area ΔOAB + area ΔOCB

+ area sector $\overset{\frown}{ADC}$

$= \frac{1}{2}\cdot 2\cdot 2\sqrt{3} + \frac{1}{2}\cdot 2\cdot 2\sqrt{3} + \frac{2}{3}\cdot\pi(2)^2$

$= \boxed{4\sqrt{3} + \frac{8}{3}\pi}$

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